PRE-CALCULUS REVIEW

FOR MATH 1500

UNIVERSITY OF MANITOBA

Contents

Preface

This booklet contains the pre-calculus material that were used during any midterm or final examination for MATH 1500 from Fall 2010 to Winter 2014. It is divided into two parts. The first part is the questions, and the second is the answers. They are separated into specific topics.

Please note that there is absolutely no guarantee that any future midterm or final exams will contain this material. Also note that every term the choice is the instructors as to which material is included from a pre-calculus course. Do not use the exams in this booklet as a list of topics covered on future exams.

The purpose of this booklet is to help you prepare for MATH 1500. Any topic in this booklet is considered knowledge that you should possess before entering a first year calculus course and you should not expect your teacher to teach you any of this material. Note that you are not permitted to use calculators for any of this material.

~~If you find any mistakes or typos, please email them to nick@nickharland.com~~

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Updated: December, 2015

Typos or mistakes can be emailed to Michelle.Davidson@umanitoba.ca

Chapter 1

Questions

1.1 Factoring

Factor the following polynomials.

1. $-x^2 + 2x + 3$
2. $x^3 - 3x^2$
3. $2x^2 - 8$
4. $3x^2 - 27$
5. $x^2 - 5x + 6$
6. $x^2 - 1$
7. $x^3 - 9x$
8. $x^2 + 2x + 1$
9. $x^3 - 6x^2 + 11x - 6$

1.2 Graphing Polynomials

Graph the following polynomials.

1. $y = -x^2 + 1$
2. $y = 3x^3 - 3x^2 - 6x$
3. $y = 2x - x^2$
4. $y = 2x^3 - 4x^2 - 6x$
5. $y = 2 - x - x^2$
6. $y = x^3 - 6x^2 + 11x - 6$

1.3 Solving Inequalities

For the following function, determine the open intervals where the function is positive and where it is negative. Further, determine the domain of the function.

1. $f(x) = \dfrac{-x-1}{(x-1)^3}$
2. $f(x) = \dfrac{2(x+2)}{(x-1)^4}$
3. $f(x) = \dfrac{2(1-x^2)}{(x^2+1)^2}$
4. $f(x) = \dfrac{4x(x^2-3)}{(x^2+1)^3}$
5. $f(x) = \dfrac{2x}{(1-x)^3}$
6. $f(x) = \dfrac{2(2x+1)}{(1-x)^4}$
7. $f(x) = \dfrac{2x^2(x-3)}{(x-2)^2}$
8. $f(x) = \dfrac{2x(x^2-6x+12)}{(x-2)^3}$
9. $f(x) = \dfrac{xe^x}{(x+1)^2}$
10. $f(x) = \dfrac{(x^2+1)e^x}{(x+1)^3}$
11. $f(x) = \dfrac{x+6}{(2-x)^3}$
12. $f(x) = \dfrac{2(x+10)}{(2-x)^4}$
13. $f(x) = -\dfrac{4(x+2)}{x^3}$
14. $f(x) = \dfrac{8(x+3)}{x^4}$
15. $f(x) = \dfrac{(x+1)(x-3)}{(x-1)^2}$
16. $f(x) = \dfrac{8}{(x-1)^3}$

1.4 Simplifying Fractions

For questions 1 to 6, simplify as much as possible.
For questions 7 to 11, simplify the absolute value and then simplify as much as possible.
For questions 12 to 17, rationalize the numerator (or denominator as in the case of 13) and then simplify as much as possible.
For questions 18 to 23, simplify until the factor of h is removed from the denominator.

1. $\dfrac{x^3 - 9x}{x - 3}$
2. $\dfrac{1}{x - 3} - \dfrac{1}{x^2 - 5x + 6}$
3. $\dfrac{3x^2 - 27}{3 - x}$
4. $\dfrac{2x^2 - 8}{x^2 - 2x}$
5. $\dfrac{-x^2 + 2x + 3}{x^3 - 3x^2}$
6. $\dfrac{\dfrac{1}{7 + x} - \dfrac{1}{7}}{x}$
7. $\dfrac{|1 - x|}{(x - 1)^2}$ if $x > 1$.
8. $\dfrac{8x + x^2}{|x|}$ for $x < 0$.
9. $\dfrac{x - 4}{|x - 4|}$ individually for $x > 4$ and $x < 4$.
10. $\dfrac{x^2 - 1}{|x - 1|}$ for $x > 1$.
11. $\sin\left(\dfrac{x^3 - x^2}{|x - 1|}\right)$ for $x < 1$.
12. $\dfrac{5 - \sqrt{x}}{25 - x}$
13. $\dfrac{x^2 - 1}{\sqrt{x + 1} - \sqrt{x^2 + 1}}$

14. $\dfrac{2\sqrt{x+6}-6}{x-3}$

15. $\dfrac{2-\sqrt{x+2}}{x-2}$

16. $\dfrac{\sqrt{x+x^2}+x}{1}$

17. $\dfrac{\sqrt{x}-1}{x-1}$

18. $\dfrac{\sqrt{1-2(x+h)}-\sqrt{1-2x}}{h}$

19. $\dfrac{\sqrt{x+h+e}-\sqrt{x+e}}{h}$

20. $\dfrac{\sqrt{(x+h)^2+9}-\sqrt{x^2+9}}{h}$

21. $\dfrac{(x+h)^2+(x+h)-x^2-x}{h}$

22. $\dfrac{1}{h}\left(\dfrac{1}{2-x-h}-\dfrac{1}{2-x}\right)$

23. $\dfrac{\sqrt{x+h-3}-\sqrt{x-3}}{h}$

1.5 Solving for one Variable

Solve the equations for Z.

1. $y\left(\frac{1}{x}\right) + Z\ln x = x\left(\frac{1}{y}\right)Z + \ln y$
2. $0 = 3x^2 + y^3 + 3xy^2Z + Z$
3. $3x^2 + 3y^2Z - 9y - 9xZ = 0$
4. $4y + 4xZ = 2(x^2 + y^2)(2x + 2yZ)$
5. $3x^2 + y + xZ + 3y^2Z = 0$
6. $2xZ + 2yY = 0$
7. $3y^2xZ + y^3 + 2x + 4Z = 0$
8. $2x + 4y + 4xZ + 2yZ = 0$
9. $3x^2 + xZ + y + 3y^2Z = 1$
10. $e^yZ + xZ + y + e^x = 0$

1.6 Word Problems

For each of the following word problems, create a function which solves for the quantity indicated. Indicate the domain.

1. A rectangular prism is designed so that it's length is always exactly twice it's height. (Volume)

2. A particle is moving along the parabola with equation $y = x^2$. (Distance from the origin)

3. A rectangular box with a square base is to be made from two kinds of materials. The material for the sides and the top costs \$10 per square metre and the material for the base costs \$20 per square metre. The volume is to be 12 m^3. (Cost to create the box)

4. A rectangular storage tank is to have a square base, an open top, and must hold 64 m^3 of water. The material to be used for the bottom cost \$4 per square meter, and the material for the sides costs \$16 per square meter. (Cost to create the box)

5. You wish to construct a rectangular open-top box with a square base. You have 48 square feet of plywood to make the sides and the bottom of the box. (Volume)

6. You are to design a can with a circular base of radius r cm and a height of h cm. Suppose that you want to decorate it with one strip of ribbon all around the circumference of the cup, and one strip along the height. There is only 30 cm of ribbon. (Volume)

7. A mathematical architect wants to construct a right triangle in the following way. One point of the triangle is $(0, 0)$, one side of the triangle is on the x-axis and the point P of the triangle is in the first quadrant and lies on the curve $y = 4 - x^2$. (Area)

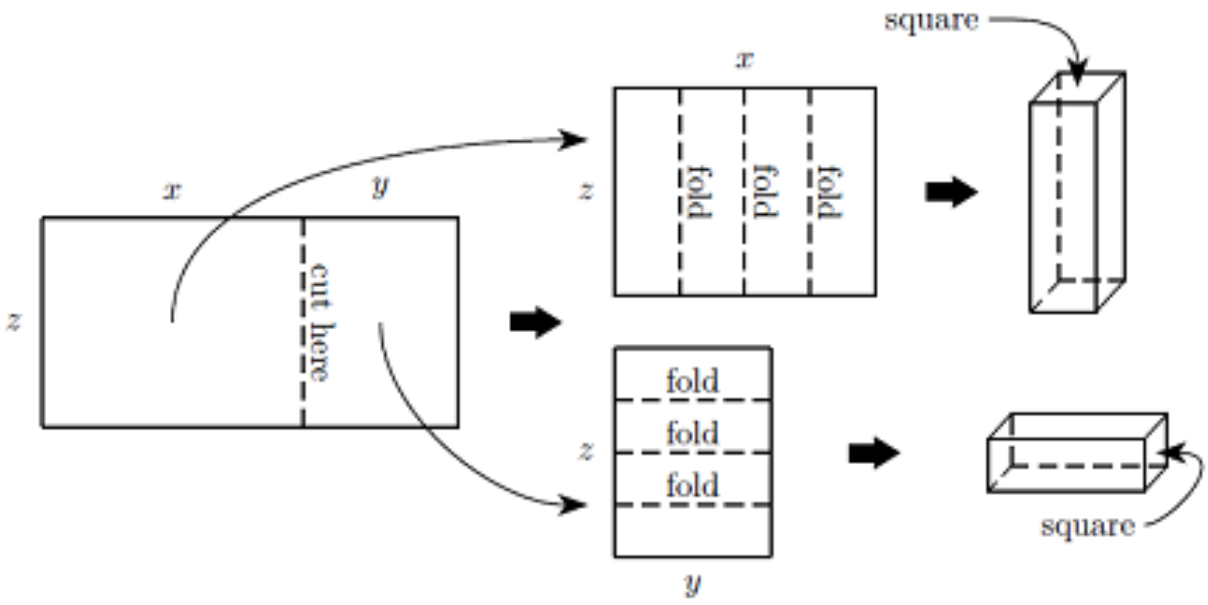
8. A Particle P is moving along the curve $y = \dfrac{1}{x}$. The line segment between the point $(0, 0)$ and P forms an angle θ between the line segment and the positive x-axis. (Any formula relating θ and x.)

9. An open top box with a square base has a volume $V = 4$ m^3. (Surface Area)

10. Find the distance of the point (x, y) on the graph of $y = \sqrt{2x}$ to the point $(4, 0)$.

11. A box with an open top is to be constructed from a square piece of cardboard with side length 12 metres. A square is to be cut out of each of the four corners of the cardboard and the sides are to be bent up to form the box. (Volume)

12. A rectangular piece of thin metal of height $z = 4$ meters and width $x + y = 8$ meters (depicted at the left of the illustration) is first cut (as illustrated), then the two parts are folded in the shape of pipes with square cross sections (as shown in the figure). (Total Volume between the two)



1.7 Logarithms

For questions 1-2, evaluate the expression.
For questions 3-8, write the function in terms of the natural logarithm, simplifying as much as possible.
For questions 9-17, simplify as much as possible.
For questions 18-20, graph the function.

1. $\ln e$
2. $e^{\ln 4}$
3. $\log_2(\tan(x))$
4. $\log_3(x^2)$
5. $\log_2(\sin x^2)$
6. $\log_\pi x$
7. $\log_5(3x)$
8. $\log_2(\cos(x))$
9. $\ln x^x$
10. $\ln(\ln x)^{1/x}$
11. $\ln x^{\ln(3x)}$
12. $\ln(2x+3)^3$
13. $\ln x^{\sec(x)}$
14. $\ln(2+\tan(x))^{\ln x}$
15. $\ln(x^3-1)^{\sin(x)}$
16. $\ln(x\sin(x))^x$
17. $\ln \dfrac{(x^2+1)^3 x^{\cos x}}{e^{2x}}$
18. $y = \ln x$
19. $y = \ln(x-1)$
20. $y = 1 - \ln x$

1.8 Trigonometry

For questions 1-8, evaluate the expression.
For questions 9-11, use an identity to evaluate the expression.
For questions 12-14, solve the equation, possibly using identities.
For questions 15-16, manipulate the expression until there are only $\sin\theta$ and $\cos\theta$ terms and any $\sin\theta$ terms must be in the form $\dfrac{\sin\theta}{\theta}$.
For questions 17-18, using identities if necessary, manipulate the expressions. For each, you should go until there is a $\dfrac{\sin\theta}{\theta}$ term and a $\dfrac{\cos\theta - 1}{\theta}$ term.
For questions 19-23, graph the function.

1. $\sin(\pi/4)$
2. $\cos(-\pi/6)$
3. $\tan(\pi/3)$
4. $\cot(11\pi/3)$
5. $\sec(-3\pi/4)$
6. $\sin(4\pi)$
7. $\sec(11\pi/2)$
8. $\csc(\pi)$

9. Solve for possible values of $\sin\theta$ if $\cos\theta = 1/3$.
10. Solve for possible values of $\tan\theta$ if $\cos\theta = -1/4$.
11. Calculate $\sin(7\pi/12)$.

12. $\sin 2x = \dfrac{1}{2}$
13. $\cos x + \sin 2x = 0$
14. $\sin x = \cos x$
15. $\dfrac{\tan x}{2x}$
16. $\sin(\pi t)\cot(11t)$
17. $\dfrac{\sin(x+h) - \sin x}{h}$
18. $\dfrac{\cos(x+h) - \cos x}{h}$
19. $y = \sin x$
20. $y = \cos x$
21. $y = \sin 2x$
22. $y = 2\sin x$
23. $y = 1 - 2\cos 3x$

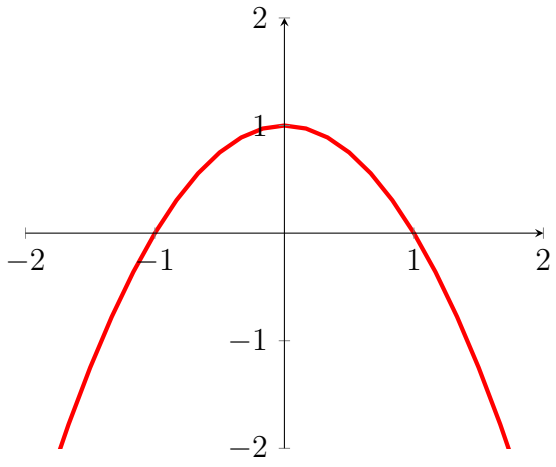
Chapter 2

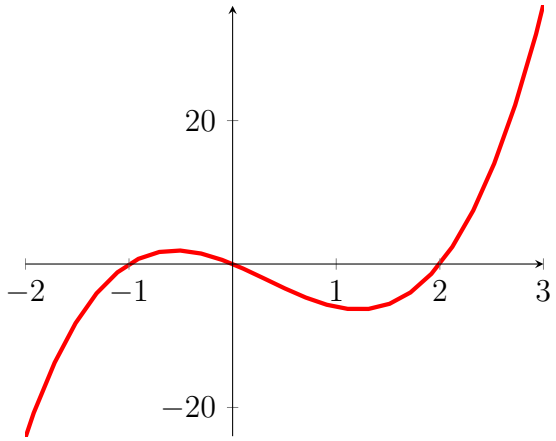
Answers

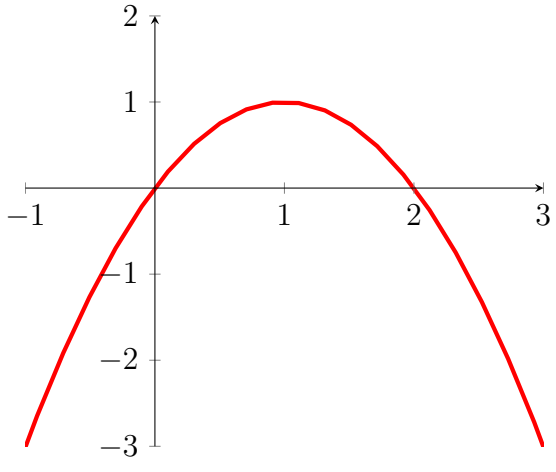
2.1 Factoring Answers

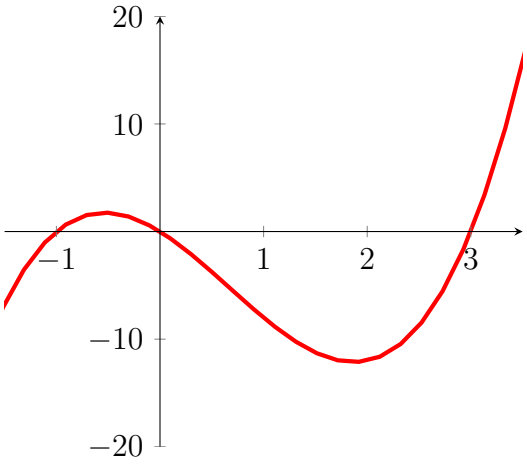
1. $(x-3)(x+1)$
2. $x^2(x-3)$
3. $2(x-2)(x+2)$
4. $3(x-3)(x+3)$
5. $(x-2)(x-3)$
6. $(x-1)(x+1)$
7. $x(x-3)(x+3)$
8. $(x+1)^2$
9. $(x-1)(x-2)(x-3)$

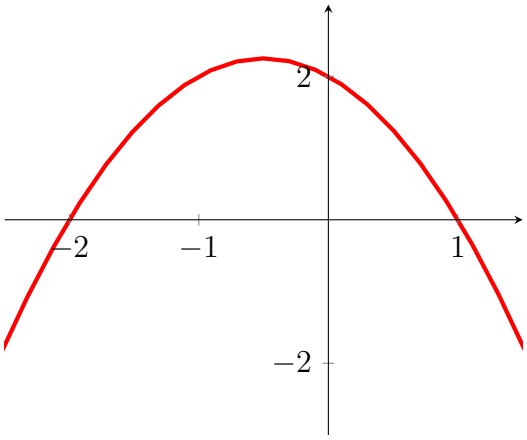
2.2 Graphing Polynomials Answers

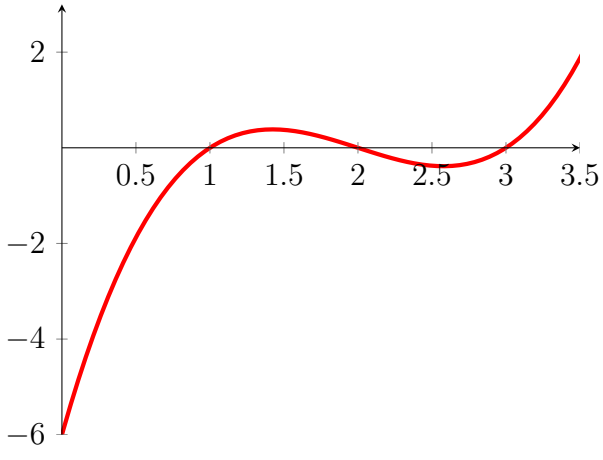
1. 

2. 

3. 

4. 

5. 

6. 

2.3 Solving Inequalities Answers

1. Positive on the interval $(-1, 1)$. Negative on the intervals $(-\infty, -1)$, $(1, \infty)$.
   The domain is $(-\infty, 1)$, $(1, \infty)$.

2. Positive on the intervals $(-2, 1)$, $(1, \infty)$. Negative on the interval $(-\infty, -2)$.
   The domain is $(-\infty, 1)$, $(1, \infty)$.

3. Positive on the interval $(-1, 1)$. Negative on the intervals $(-\infty, -1)$, $(1, \infty)$.
   The domain is $(-\infty, \infty)$.

4. Positive on the intervals $(-\sqrt{3}, 0)$, $(\sqrt{3}, \infty)$.
   Negative on the intervals $(-\infty, -\sqrt{3})$, $(0, \sqrt{3})$. The domain is $(-\infty, \infty)$.

5. Positive on the interval $(0, 1)$. Negative on the intervals $(-\infty, 0)$, $(1, \infty)$.
   The domain is $(-\infty, 1)$, $(1, \infty)$.

6. Positive on the intervals $(-1/2, 1)$, $(1, \infty)$. Negative on the interval $(-\infty, -1/2)$.
   The domain is $(-\infty, 1)$, $(1, \infty)$.

7. Positive on the interval $(3, \infty)$. Negative on the intervals $(-\infty, 0)$, $(0, 2)$, $(2, 3)$.
   The domain is $(-\infty, 2)$, $(2, \infty)$.

8. Positive on the intervals $(-\infty, 0)$, $(2, \infty)$. Negative on the interval $(0, 2)$.
   The domain is $(-\infty, 2)$, $(2, \infty)$.

9. Positive on the interval $(0, \infty)$. Negative on the intervals $(-\infty, -1)$, $(-1, 0)$.
   The domain is $(-\infty, -1)$, $(-1, \infty)$.

10. Positive on the interval $(-1, \infty)$. Negative on the interval $(-\infty, -1)$.
    The domain is $(-\infty, -1)$, $(-1, \infty)$.

11. Positive on the interval $(-6, 2)$. Negative on the intervals $(-\infty, -6)$, $(2, \infty)$.
    The domain is $(-\infty, 2)$, $(2, \infty)$.

12. Positive on the intervals $(-10, 2)$, $(2, \infty)$. Negative on the interval $(-\infty, -10)$.
    The domain is $(-\infty, 2)$, $(2, \infty)$.

13. Positive on the intervals $(-\infty, -2)$, $(0, \infty)$. Negative on the interval $(-2, 0)$.
    The domain is $(-\infty, 0)$, $(0, \infty)$.

14. Positive on the interval $(-\infty, -3)$. Negative on the intervals $(-3, 0)$, $(0, \infty)$.
    The domain is $(-\infty, 0)$, $(0, \infty)$.

15. Positive on the intervals $(-\infty, -1)$, $(3, \infty)$.
    Negative on the intervals $(-1, 1)$, $(1, 3)$. The domain is $(-\infty, 1)$, $(1, \infty)$.

16. Positive on the interval $(1, \infty)$. Negative on the interval $(-\infty, 1)$.
    The domain is $(-\infty, 1)$, $(1, \infty)$.

2.4 Simplifying Fractions Answers

1. $x(x+3)$
2. $\dfrac{1}{x-2}$
3. $-3(x+3)$
4. $\dfrac{2(x+2)}{x}$
5. $\dfrac{-(x+1)}{x^2}$
6. $\dfrac{-1}{7(7+x)}$
7. $\dfrac{1}{x-1}$
8. $-8-x$
9. For $x > 4$ it is 1. For $x < 4$ it is -1.
10. $x+1$
11. $\sin(-x^2)$
12. $\dfrac{1}{5+\sqrt{x}}$
13. $\dfrac{-(x+1)(\sqrt{x+1}+\sqrt{x^2+1})}{x}$
14. $\dfrac{2}{\sqrt{x+6}+3}$
15. $\dfrac{-1}{2+\sqrt{x+2}}$
16. $\dfrac{x}{\sqrt{x+x^2}-x}$
17. $\dfrac{1}{\sqrt{x}+1}$
18. $\dfrac{-2}{\sqrt{1-2(x+h)}+\sqrt{1-2x}}$

19. $\dfrac{1}{\sqrt{x+h+e}+\sqrt{x+e}}$

20. $\dfrac{2x+h}{\sqrt{(x+h)^2+9}+\sqrt{x^2+9}}$

21. $2x+h+1$

22. $\dfrac{1}{(2-x-h)(2-x)}$

23. $\dfrac{1}{\sqrt{x+h-3}-\sqrt{x-3}}$

2.5 Solving for one Variable Answers

1. $Z = \dfrac{\ln y - y/x}{\ln x - x/y}$
2. $Z = \dfrac{-3x^2 - y^3}{3xy^2 + 1}$
3. $Z = \dfrac{3y - x^2}{y^2 - 3x}$
4. $Z = \dfrac{y - x^3 - xy^2}{-x + x^2y + y^3}$
5. $Z = \dfrac{-3x^2 - y}{x + 3y^2}$
6. $Z = \dfrac{-y}{x}$
7. $Z = \dfrac{-y^3 - 2x}{3y^2x + 4}$
8. $Z = \dfrac{-x - 2y}{2x + y}$
9. $Z = \dfrac{1 - 3x^2 - y}{x + 3y^2}$
10. $Z = \dfrac{-y - e^x}{e^y + x}$

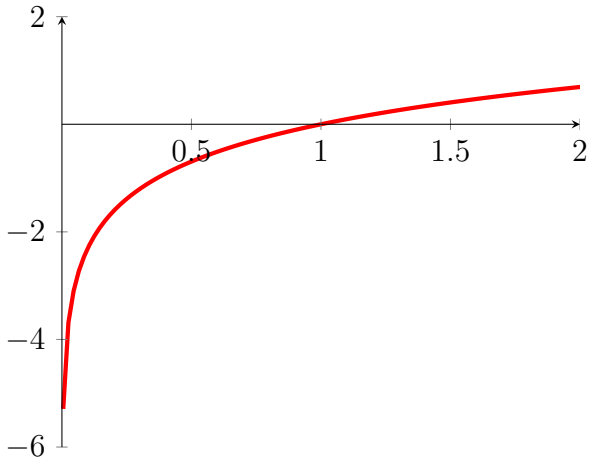
2.6 Word Problems Answers

1. $V(w, h) = 2wh^2$. w and h have domain $(0, \infty)$.
2. $d(x) = \sqrt{x^2 + x^4}$. Domain: $(-\infty, \infty)$.
3. $C(x) = 30x^2 + \dfrac{480}{x}$. Domain: $(0, \infty)$.
4. $C(x) = 4x^2 + \dfrac{4096}{x}$. Domain: $(0, \infty)$.
5. $C(x) = 12x - \dfrac{x^3}{4}$. Domain: $(0, \sqrt{48})$.
6. $V(x) = 30\pi r^2 - 2\pi^2 r^3$. Domain: $\left(0, \dfrac{15}{\pi}\right)$.
7. $A(x) = 2x - \dfrac{x^3}{2}$. Domain: $(0, 2)$.
8. Many answers. For example. $\tan\theta = \dfrac{1}{x^2}$, or $x^2 = \cot\theta$. Domain for θ: $(0, \pi/2)$. Domain for x: $(0, \infty)$.
9. $V(x) = x^2 + \dfrac{16}{x}$. Domain: $(0, \infty)$.
10. $d(x) = \sqrt{x^2 - 6x + 16}$. Domain: $[0, \infty)$.
11. $V(x) = 4x^3 - 48x^2 + 144x$. Domain: $(0, 6)$.
12. $V(x) = \dfrac{x^2}{4} - x + 8$. Domain: $(0, 8)$.

2.7 Logarithms Answers

1. 1
2. 4
3. $\dfrac{\ln(\tan(x))}{\ln 2}$
4. $\dfrac{2\ln x}{\ln 3}$
5. $\dfrac{\ln(\sin x^2)}{\ln 2}$
6. $\dfrac{\ln x}{\ln \pi}$
7. $\dfrac{\ln 3 + \ln x}{\ln 5}$
8. $\dfrac{\ln(\cos x)}{\ln 2}$
9. $x \ln x$
10. $\dfrac{\ln(\ln x)}{x}$
11. $\ln(3x) \ln x$
12. $3\ln(2x+3)$
13. $\sec x \ln x$
14. $\ln x \ln(2 + \tan(x))$
15. $\sin x \ln(x^3 - 1)$
16. $x\big(\ln x + \ln(\sin x)\big)$
17. $3\ln(x^2+1) + \cos x \ln x - 2x$

18.



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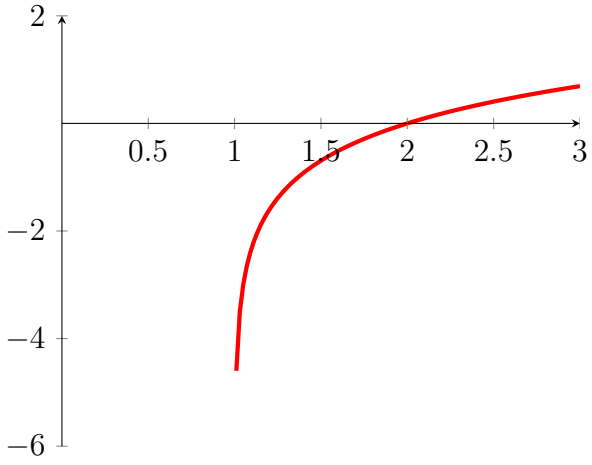

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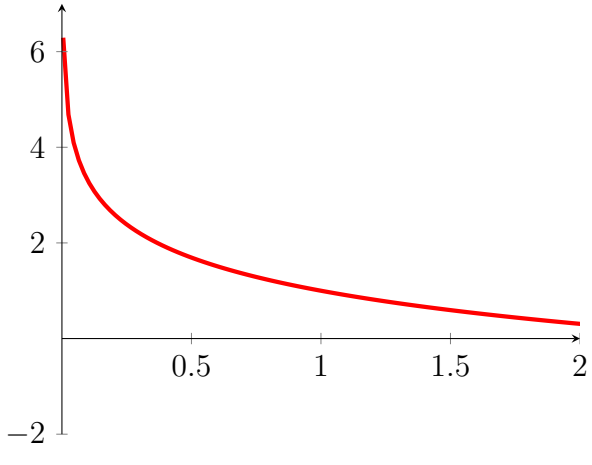

2.8 Trigonometry Answers

1. $\frac{1}{\sqrt{2}}$ or $\frac{\sqrt{2}}{2}$
2. $\frac{\sqrt{3}}{2}$
3. $\sqrt{3}$
4. $-\frac{1}{\sqrt{3}}$
5. $-\sqrt{2}$
6. 0
7. undefined
8. undefined
9. $\pm\frac{\sqrt{8}}{3}$
10. $\pm\sqrt{15}$
11. $\frac{\sqrt{6}+\sqrt{2}}{4}$
12. $x = \frac{\pi}{12} + k\pi,\ \frac{11\pi}{12}$ where k is any integer.
13. $x = \frac{\pi}{2} + 2k\pi,\ \frac{3\pi}{2} + 2k\pi,\ \frac{5\pi}{6} + 2k\pi,\ \frac{7\pi}{6} + 2k\pi$ where k is any integer.
14. $x = \frac{\pi}{4} + 2k\pi,\ \frac{3\pi}{4} + 2k\pi,\ \frac{5\pi}{4} + 2k\pi,\ \frac{7\pi}{4} + 2k\pi$ where k is any integer.
15. $\left(\frac{\sin x}{x}\right)\left(\frac{1}{2\cos x}\right)$
16. $\left(\frac{\sin(\pi t)}{\pi t}\right)\left(\frac{1}{\frac{\sin(11t)}{11t}}\right)\left(\frac{\pi\cos(11t)}{11}\right)$
17. $\sin x\left(\frac{\cos h - 1}{h}\right) + \cos x\left(\frac{\sin h}{h}\right)$

18. $\cos x\left(\dfrac{\cos h - 1}{h}\right) - \sin x\left(\dfrac{\sin h}{h}\right)$

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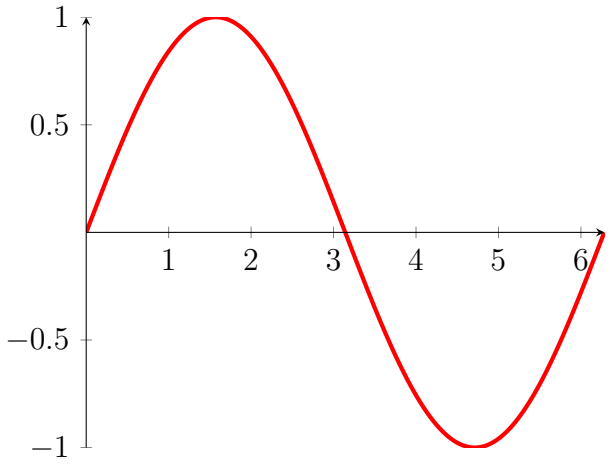

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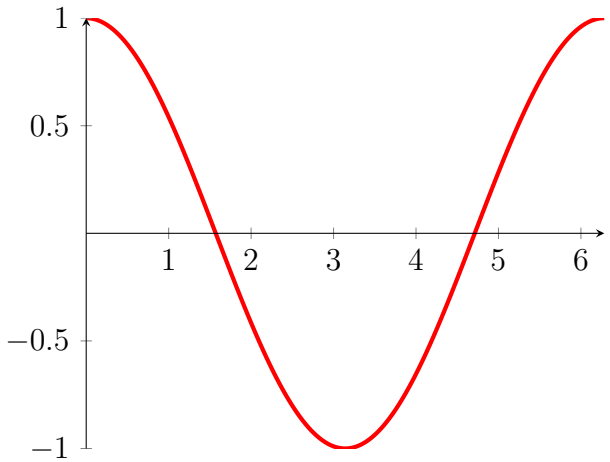

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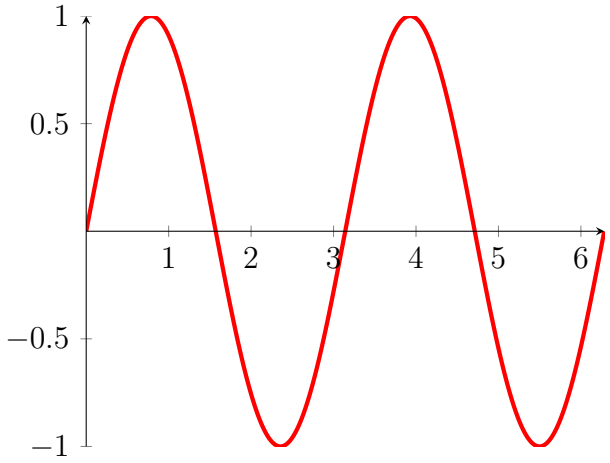
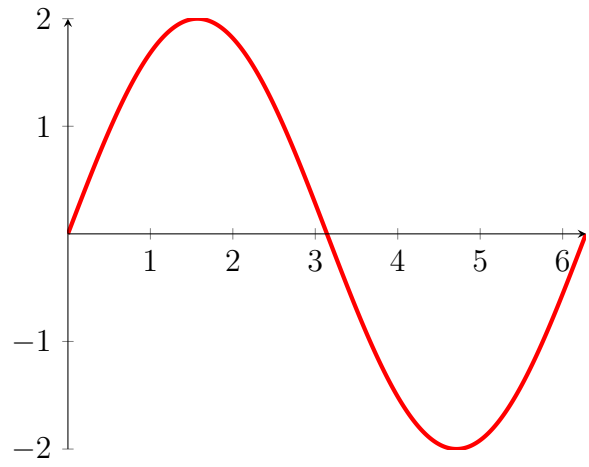


22. 

23.